Lecture 22 (3/2/22)

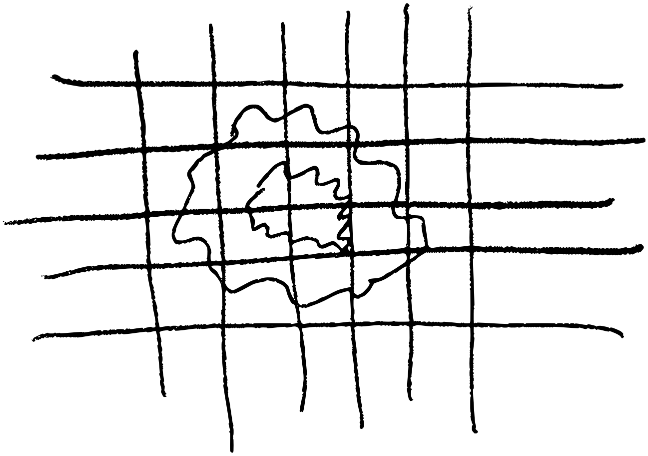
<u>Runge's Thm.</u> Let K be compact, $E \subseteq \mathbb{C}_\infty \setminus K$ a set that intersects each component of $\mathbb{C}_\infty \setminus K$. If $K \subset\subset G$, $f \in H(G)$, then f can be unif. approx. on K by rational fcns w/ poles in E.

<u>Pf.</u> We shall follow the outline indicated in last lecture.

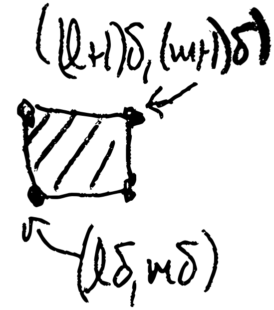
① <u>Prop 1.</u> There exist a finite number of line segments $\gamma_1, \dots, \gamma_n$ in $G \setminus K$ s.t. for $z \in K$:

$$f(z) = \sum_{k=1}^{n} \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(\zeta)\,d\zeta}{\zeta - z}.$$

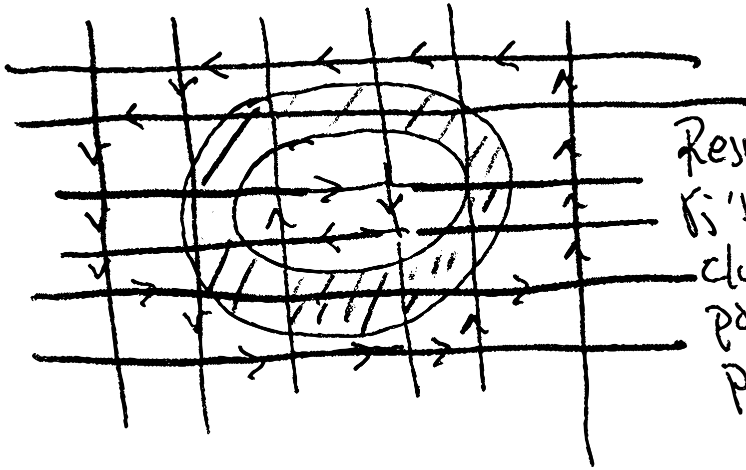
<u>Pf.</u> Let $\delta > 0$ s.t. $\sqrt{2}\,\delta < d(K, \mathbb{C} \setminus G)$. Consider the mesh of vertical and horizontal lines $\operatorname{Re} z = l\delta$, $\operatorname{Im} z = m\delta$, $l, m \in \mathbb{Z}$.



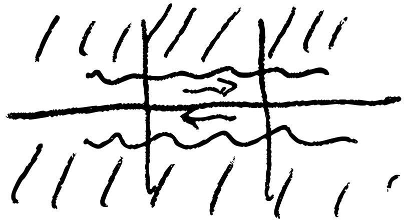
Let $R_{\ell m}$ denote closed square



Let $A = \{(\ell,m) : R_{\ell m} \cap K \neq \emptyset\}$. A is finite by compactness and all $R_{\ell m} \subset\subset G$ for $(\ell,m) \in A$, by def. of δ. Let $\gamma_1, \dots, \gamma_n$ be the line segments of $\partial R_{\ell m}$, $(\ell,m) \in A$, s.t. $\gamma_j \subset\subset G \setminus K$ and and pos. oriented w.r.t. $R_{\ell m}$. Since each line segment in the boundary of a square belongs to precisely two squares, some line segments γ_i, γ_j could be the same but w/ opposite orientation. Delete those.



Resulting γ_i's forming closed polygonal paths.



γ_i, γ_j cancel.

Now, if we pick $z \in K$ there are 3 cases:

(i) $z \in \operatorname{int} R_{l_0,m_0}$ for some $(l_0, m_0) \in A$

Then $n(z, \partial R_{l_0 m_0}) = 1$ and

$n(z, \partial R_{jk}) = 0$ for all other (j,k).

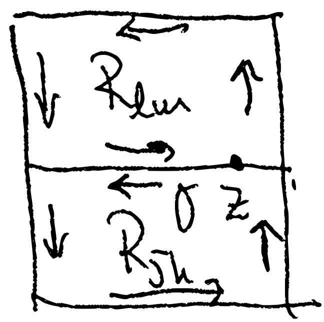
By CIF

$$f(z) = \sum_{(l,m)\in A} \frac{1}{2\pi i} \int_{\partial R_{lm}} \frac{f(\zeta)}{\zeta - z}\, d\zeta =$$

$$= \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$ after deleting

line segments that are integrated over in both directions.

(ii) $z \in \mathring{\gamma}$ ← open line segment bounding R_{lm} and R_{jk}.



Then γ is <u>not</u> one of the $\gamma_1, \dots, \gamma_n$

Cannot apply CIF w/ $z \in \gamma$, but we can delete γ and form $\tilde{R} = R_{lm} \cup R_{jk}$. Then $z \in \operatorname{int} \tilde{R}$ and we can proceed as in (i)

(iii) $z \in$ vertex on boundary of 4 squares. Merge all 4 into one and proceed as in (i). Details are DIY.

The point is that in all 3 cases, the line segments $\gamma_1, \ldots, \gamma_n$ remain the same since those removed in (ii) and (iii) are not among those selected up front. This completes Pf of Prop 1. $\boxtimes$

② Approximate $\frac{1}{2\pi i}\int_{\gamma_j} \frac{f(\zeta)}{\zeta - z}\,d\zeta$ by

rational functions w/ poles in E.
Let us fix $\gamma = \gamma_j$ for some $j \in \{1, \dots, n\}$. First:

Prop 2. $\forall \varepsilon > 0$ $\exists$ rational fcn $R(z)$ w/ poles on $\gamma \subset\subset G \setminus K$ s.t.

$$\sup_K \left| \frac{1}{2\pi i}\int_\gamma \frac{f}{\zeta - z}\,d\zeta - R \right| < \varepsilon$$

Pf. This is clear by using Riemann sums, since

$$\int_\gamma \frac{f(\zeta)\,d\zeta}{\zeta - z} = \int_0^1 \frac{f(\gamma(t))\,\gamma'(t)\,dt}{\gamma(t) - z}$$

So, take a sufficiently fine partition $0 = t_0 < t_1 < \dots < t_N = 1$ and let

$$R(z) = \sum_{j=1}^{N} \frac{f(\gamma(t_j))\,\gamma'(t_j)}{\gamma(t_j) - z}(t_j - t_{j-1})$$

The conclusion holds by def. of Riemann integral and uniform continuity of integrand. Details are DIY. ☒

The rational functions so constructed have simple poles at points $a_j = \gamma(t_j)$ on $\gamma \subset\subset G \setminus K$. To approximate by rational functions w/ poles in E, it suffices (moments reflection) to approximate $\frac{1}{z-a}$, $a \in \mathbb{C} \setminus K$, unif. on K by rational functions in E.

<u>Prop 3.</u> For $a \in \mathbb{C} \setminus K$, $f(z) = \frac{1}{z-a}$ can be unif. approximated on K by rational functions w/ poles in E.

Pf. Let $B \subseteq \mathbb{C} \setminus K$ be the set of a s.t. conclusion holds. Clearly $E \subseteq B$. Let U be a component of $\mathbb{C} \setminus K$. First, assume U is not the unbdd component. Since $E \cap U \neq \emptyset$ $\Rightarrow B \cap U \neq \emptyset$. We show $B \cap U$ open + closed $\Rightarrow U \cap B = U$ (as desired).

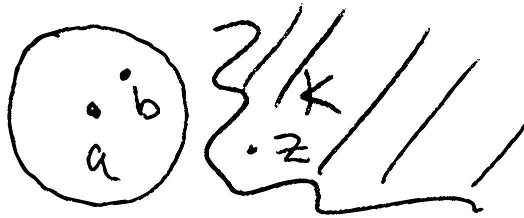
- <u>$B \cap U$ closed.</u> This is clear since if $a_j \in B$, $a_j \to a \in U$ then

$$\frac{1}{z-a_j} \to \frac{1}{z-a} \text{ unif. on } K$$

and since we can find $R_{jk}(z)$ w/ poles in E s.t. $R_{jk} \xrightarrow[k\to\infty]{} \frac{1}{z-a_j}$ unif. on K, $\exists R_{jk_j} \to \frac{1}{z-a}$ unif on K $\Rightarrow$ $a \in B \cap U$

- <u>$B \cap U$ open</u>: Let $a \in B \cap U$, $\overline{B(a,\varepsilon)} \subseteq U$,

$b \in B(a,\varepsilon)$:



We have $d(K, \overline{B(a,\varepsilon)}) = \varepsilon' > \varepsilon \Rightarrow$

$|b-a| < \varepsilon < \varepsilon' \leq |z-a| \quad \forall z \in K$:

$$\Rightarrow \left|\frac{b-a}{z-a}\right| < \delta = \frac{\varepsilon}{\varepsilon'} < 1.$$

But then $$\frac{1}{z-b} = \frac{1}{z-a+a-b} =$$

$$\frac{1}{z-a} \cdot \frac{1}{1-\frac{b-a}{z-a}} = \frac{1}{z-a} \sum_{k=0}^{\infty} \left(\frac{b-a}{z-a}\right)^k$$

w/ unif. conv. on K. Thus,

$R_n = \frac{1}{z-a} \sum_{n=0}^{n} \left(\frac{b-a}{z-a}\right)^k$ is a seq.

of rat. fcns w/ poles at $z=a$.

Since $a \in B$, we can approximate the R_n by rational functions w/ poles on $\bar{E}$ and, as above, this gives approx. of $\frac{1}{z-b}$ by rat. fcns w/ poles on $\bar{E}$. $\Rightarrow$

$b \in B$. $\Rightarrow$ $B \cap \bar{U}$ open.

This shows $U \cap B = \bar{U}$ as desired.

If U is unbounded component then the same argument works if E contains a point in U (not only ∞). What if E's intersection w/ unbdd component U of $\mathbb{C}_\infty \setminus K$ is only ∞? By the argument above, we may approximate $\frac{1}{z-a}$, $a \in U$ by rational functions

with a pole only at $b \in \overline{U}$ and we may assume $K \subset\subset B(0, |b|)$.
But then, $\exists\, R < |b|$ s.t. $K \subset\subset B(0,R)$ and rational functions w/ pole at b are analytic in $B(0,R)$.
Thus, such rational fcns can be approximated by polynomials (by Taylor expansion), a.k.a. rational functions w/ pole only at ∞. As above, polynomials then approximate $\frac{1}{z-a}$ for all $a \in \overline{U}$. This completes the pf of Prop 3. $\boxtimes$
The pf of Runge now follows by putting together Prop's 1–3. DIY. $\boxtimes$